

Spherical harmonics for the compactification of IIB supergravity on S_5

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We give a detailed derivation of the spherical harmonics which correspond to the spectrum in the tables of the 1985 paper by Kim, Romans and the author on the Kaluza-Klein dimensional reduction of IIB supergravity on S_5 to 5-dimensional maximal anti de Sitter supergravity. We show how all methods that have been used in the literature to obtain spectra if the internal space is S_n can be derived from embedding S_n in \mathbb{R}^{n+1} . A crucial observation for spin 1/2 spherical harmonics is that one also needs a local Lorentz rotation if one transforms to polar or stereographic coordinates. The relation to the vector spherical harmonics of electromagnetism is also worked out.

I. INTRODUCTION

In the 1980's many physicists considered the Kaluza-Klein (KK) compactification of supergravity models and worked out their spectra (the values of the masses and their degeneracies). One such article[1] dealt with the compactification of IIB supergravity on S_5 , yielding maximal ($N = 8$) supergravity in 5-dimensional anti de Sitter spacetime. The authors were H. J. Kim who was a graduate student at Stony Brook, Larry Romans who was a graduate student at CalTech, and the author. We denote this 1985 paper by KRN in what follows. The article contains a series of tables and figures for the spectrum. Many people have used it for the AdS/CFT program of string theory, and one person (Ian Kogan) told me long ago that he had checked all formulas on the computer. However, the article is rather condensed and I have often been asked for more explanation and more background material. I always demurred because in KRN one particular method was used but I wanted first to understand how it was related to other methods which were also widely used by other physicists. In the course of teaching string courses at Stony Brook the answers came slowly to me, so that I now believe I can present a clear picture. The aim of this article is thus pedagogical: a discussion of the various methods to obtain spectra on S_n , and their interrelations. Experts in KK theory have each their own methods, and may not even want to spend time on reading about other method. We shall derive explicit expressions for the spherical harmonics (in particular Killing spinors) but such information is not necessary if one is only interested in the spectra. In the conclusions we shall present a very quick method to obtain spectra on general coset manifolds G/H : all one needs to know are the Casimir operators of G and H .

There exist excellent reviews on KK theory[2], where complete lists of references can be found. In this review we shall only refer to those articles that form the basis of our approach and with which we are thoroughly familiar. Conference proceedings (as opposed to talks

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at conferences) should contain reviews that remain of interest as time goes on. Since Max Kreuzer also spent much time in updating his research in lectures, I decided to publish this material in these proceedings.

Let me now first write a few words about Max. Max Kreuzer carried String Theory at the TU (and, one might add, in all of Austria). Through Wolfgang Kummer, who died in 2007, I have been visiting the TU every Jänner (Jänner is January in Austrian) for the past 12 years, and I developed a friendship with Wolfgang, Toni Rebhan and Max. Every day we would have lunch together and drink coffee afterwards in a large office of graduate students. A few times we went to the famous TU ball in the imperial Hofburg, where Max would disappear in the ballroom for salsa dancing and be lost for hours. During my visits he got interested in work I had done with Antonio Grassi on the BRST quantization of the Berkovits superstring, and with Sebastian Guttenberg and Johanna Knapp they wrote an article (On the covariant quantization of type II superstring”, hep-th/0405007). They did not solve the problem, nor did we, but their article is quite interesting.

Twice Max helped me with solving a problem I had for a long time. One problem I encountered while writing a book on Advanced Quantum Gauge Field Theory. One can give a clear physical explanation of the asymptotic freedom of QCD if one works in the Coulomb gauge. However, calculations in the Coulomb gauge are tedious, even one-loop calculations. To my pleasant surprise I found an article by Max and Wolfgang which contained these calculations (Nucl. Phys. B **276** (1986) 466) but in very abridged form. Max went home, found his old calculations, and pieced together the explicit one-loop results I needed.

Another problem I ran into while teaching string theory. The spectrum of the closed bosonic (or spinning) string is defined by the BRST condition $(Q_L + Q_R)|phys\rangle = 0$, but the result is the direct product of $Q_L|phys\rangle = 0$ and $Q_R|phys\rangle = 0$. This fact is known as the Künneth formula in cohomology theory, but the proofs I found in math books were rather complicated. One day I explained to Max my frustration at finding a simple proof, and the next day he came back with an extremely simple proof, just based on linear algebra. I now use it in my string lectures.

Max taught superb classes on string theory whose (rather condensed) lecture notes one can find on the web[3]. I followed several of his classes, and each time I felt sorry I had to leave by the end of Jänner. With fond memory to Max and his classes, I dedicate this lecture to the good and interesting times we had together.

II. SCALAR SPHERICAL HARMONICS ON S_n

The construction of spherical harmonics for scalar fields on S_n is well-known, but we present it here as an introduction to the more complicated cases of higher spin which we discuss later. In the Kaluza-Klein dimensional reduction, a scalar field $B(x, z)$ in D -dimensional Minkowski space with coordinates (x, z) is expanded in terms of harmonics on the compact space with coordinates z in the following manner

$$B(x, z) = \sum_{I_1} B_{I_1}(x) Y^{I_1}(z) \quad (1)$$

The $Y^{I_1}(z)$ are eigenfunctions of the d'Alembertian on S_n , which we shall denote by \square_S , and $B_{I_1}(x)$ are fields in d -dimensional (usually anti de Sitter) space with coordinates x . We shall no longer consider these fields B_{I_1} and coordinates x in what follows, but concentrate on

$Y^{I_1}(z)$. The symbol I_1 indicates that these fields have one component on S_n . For example, vector harmonics on S_5 are denoted in KRN by $Y_\alpha^{I_5}$ if they are transversal ($D^\alpha Y_\alpha^{I_5} = 0$) and by $D_\alpha Y^{I_1}$ if they are longitudinal (not transversal). Here D_α is the covariant derivative on S_n .

A word of caution. In KRN the D -dimensional field equations are reduced on S_5 to field equations of the form

$$(\square_x + \text{more})\varphi(x)Y(z) = (\square_z + \text{more})\varphi(x)Y(z) \quad (2)$$

where the “+more” on the left-hand side is part of the field equations in d dimensions, and the “+more” on the right-hand side is part of the mass operator. (Actually, mixing occurs, but never more than 2×2 mixing, and one can rather easily diagonalize the field equations.) The parts “+more” are easy to evaluate because the Riemann and Ricci tensors can be expressed in terms of the metric (as always for a maximally symmetric space). The hard problem is the determination of the eigenvalues and their degeneracies of \square_z . This is the problem discussed in this article. Due to the “+more” in the mass operator, the figures in KRN are shifted with respect to the figures we construct below, but the degeneracies are the same. Also, in KRN the mass spectrum of all scalars in d dimensions is plotted in one figure, but these scalars correspond to scalar, vector, and tensor harmonics.

What are the $Y^{I_1}(z)$ in (1)? The $Y_{lm}(\theta, \varphi)$ of quantum mechanics suggest an answer.

$$\left. \begin{aligned} Y_{0,0} &= 1; & Y_{1,\pm 1} &\sim \sin \theta e^{\pm i\varphi} = x \pm iy \\ Y_{1,0} &\sim \cos \theta = z; & Y_{2,0} &\sim 3 \cos^2 \theta - 1 = 2z^2 - x^2 - y^2 \\ Y_{2,\pm 1} &\sim (x \pm iy)z; & Y_{2,\pm 2} &\sim (x + iy)^2 \end{aligned} \right\} \quad \begin{array}{l} \text{all at} \\ r = 1 \end{array} \quad (3)$$

These are all homogeneous traceless polynomials in x, y, z at radius $r = 1$. We therefore begin with a homogeneous polynomial in \bar{x}^μ where \bar{x}^μ ($\mu = 1, \dots, n+1$) are the Cartesian coordinates of the flat embedding space \mathbb{R}^{n+1} . We can formally write these polynomials as [4, 5]

$$\bar{P}^{(l)}(\bar{x}) = c_{\mu_1 \dots \mu_l} \bar{x}^{\mu_1} \dots \bar{x}^{\mu_l} \quad (4)$$

We require that $\bar{\square} \bar{P} = 0$ where $\bar{\square} = \frac{\partial}{\partial \bar{x}^\mu} \frac{\partial}{\partial \bar{x}^\nu} \delta^{\mu\nu}$. Then c is traceless, $\delta^{\mu\nu} c_{\mu\nu\mu_3 \dots \mu_l} = 0$ (recall $Y_{2,0} \sim 2z^2 - x^2 - y^2$). We go over to polar coordinates $\hat{y}^\mu = (r, \theta^\alpha)$ where r is radius of S_n and θ^α ($\alpha = 1, \dots, n$) are any angles on S_n . The caret $\hat{}$ will always denote quantities in the system with coordinates \hat{y} . The metric in polar coordinates is of the form

$$\hat{g}_{\mu\nu}(\hat{y}) = \begin{pmatrix} 1 & 0 \\ 0 & r^2 g_{\alpha\beta}(\theta) \end{pmatrix} \quad (5)$$

Then $\hat{g} = \det \hat{g}_{\mu\nu} = r^{2n} g(\theta)$, and

$$\bar{\square} = \hat{\square} = \frac{1}{\sqrt{\hat{g}}} \frac{\partial}{\partial \hat{y}^\mu} \sqrt{\hat{g}} \hat{g}^{\mu\nu} \frac{\partial}{\partial \hat{y}^\nu} = \frac{1}{r^n} \partial_r r^n \partial_r + \frac{1}{r^2} \square_S(\theta) \quad (6)$$

where $\square_S(\theta)$ is the d'Alembertian on S_n . Since $\hat{P}^{(l)}(\bar{x})$ is homogeneous in \bar{x} , it factorizes

$$\bar{P}^{(l)}(\bar{x}) = \hat{P}^{(l)}(\hat{y}) = r^l Y(\theta) \quad (7)$$

and substituting this expression into $\bar{\square}\bar{P}^{(l)} = 0$ one arrives at

$$\bar{\square}\bar{P}^{(l)}(\bar{x}) = \hat{\square}\hat{P}^{(l)}(\hat{y}) = \left[\frac{1}{r^2}l(l+n-1) + \frac{1}{r^2}\square_S(\theta) \right] r^l Y(\theta) = 0 \quad (8)$$

Hence the eigenvalues λ_s of $-\square_S(\theta)$ on S_n are given by

$$\lambda_s(n, l) = l(n + l - 1); \quad l = 0, 1, 2, \dots \quad (s \text{ for scalar}) \quad (9)$$

On S_2 with the usual Y_{lm} one finds the familiar result $\lambda_s = l(l+1)$ of quantum mechanics, and on S_1 one finds $\lambda_s = l^2$, the 1926 result of Klein and Fock. On S_5 we find $\lambda(5, l) = l(l+4)$.

To determine the degeneracy $d_s(n, l)$ of these eigenvalues, we note[4, 5] that it is given by the number of polynomials $\bar{P}^{(l)}(\bar{x})$, that is, it is equal to the number of symmetric polynomials of order l in $n + 1$ dimensions, minus the number of such polynomials of order $l - 2$ (the trace)

$$d_s(n, l) = \binom{n+l}{l} - \binom{n+l-2}{l-2} \quad (10)$$

The index I_1 in (1) labels these harmonics, see (3). On S_2 one finds $d_s = 2l + 1$, and on S_1 one finds $d_s = 2$ except for the massless mode with $l = 0$ which is not degenerate. On S_5 , the case of interest, we find the following spectrum

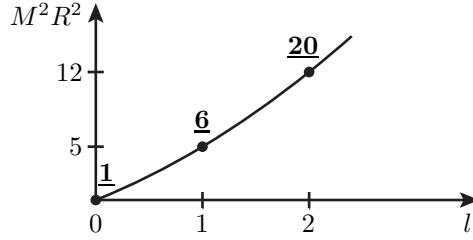


Fig. 1. S_5 , scalars (cf. Figure 2 of KRN)

There is one harmonic with vanishing eigenvalue, the constant, then 6 harmonics \bar{x}^μ , and 20 harmonics $\bar{x}^\mu \bar{x}^\nu - \frac{1}{6} \delta^{\mu\nu} \bar{x}^2$, etc. In Figure 2 of KRN one finds several curves for different x -space scalars whose masses are due to different harmonics (scalar, vector, tensor harmonics), but one of these curves corresponds to scalar harmonics and agrees with our Figure 1.

Another popular set of coordinates on S_n are the stereographic coordinates $\hat{y}^\nu = (z^\alpha, R)$ with $\alpha = 1, \dots, n$ and R the radius of S_n (we write R instead of r to indicate that we are using stereographic coordinates). The metric in stereographic coordinates is not diagonal

$$\hat{g}^{\mu\nu} = \begin{pmatrix} \left(\frac{4R^2 + z^2}{4R^2} \right)^2 \delta^{\alpha\beta} + \frac{z^\alpha z^\beta}{R^2} & z^\alpha / R \\ z^\beta / R & 1 \end{pmatrix} \quad (11)$$

as follows from squaring the vielbein field we obtain in the section on spin 1/2 spherical harmonics (see equation (62)). We discuss spherical harmonics in stereographic coordinates further at the end of that section. They give, of course, the same spectrum as polar coordinates.

III. VECTOR SPHERICAL HARMONICS

The transversal vector spherical harmonics $Y_\alpha(\theta)$ on S_n satisfying $D^\alpha(\theta)Y_\alpha(\theta) = 0$ are obtained[4, 5] from vector fields $\bar{P}_\mu^{(l)}(\bar{x})$ whose components $\mu = 1, \dots, n+1$ are homogeneous polynomials in \bar{x}^μ . We impose the equation $\square \bar{P}_\mu^{(l)} = 0$ which implies that each component is traceless. We choose a basis in which only one component of $\bar{P}_\mu^{(l)}$ is nonzero, and we call this the μ -th component. For example

$$\begin{pmatrix} 0 \\ 0 \\ 2z^2 - x^2 - y^2 \end{pmatrix} \quad \text{has} \quad \begin{array}{l} n = 2 \\ l = 2 \\ \mu = 3 \end{array} \quad (12)$$

and is an example of $\bar{P}_\mu^{(l)}(\bar{x})$.

We go over to polar coordinates $\hat{y}^\nu = (r, \theta^\alpha)$ with $\alpha = 1, \dots, n$, and by substitution and the chain rule we obtain

$$\square \bar{P}_\mu^{(l)} = \frac{\partial \hat{y}^\nu}{\partial \bar{x}^\mu} \hat{\square} \hat{P}_\nu^{(l)} = 0 \quad \Rightarrow \quad \hat{\square} \hat{P}_r^{(l)} = 0 \quad \text{and} \quad \hat{\square} \hat{P}_\alpha^{(l)} = 0 \quad (13)$$

where

$$\begin{aligned} \hat{P}_r^{(l)}(\hat{y}) &= \frac{\partial \bar{x}^\mu}{\partial r} \bar{P}_\mu^{(l)}(\bar{x}) = \frac{\bar{x}^\mu}{r} \bar{P}_\mu^{(l)}(\bar{x}) = r^l \rho(\theta) \\ \hat{P}_\alpha^{(l)} &= \frac{\partial \bar{x}^\mu}{\partial \theta^\alpha} \bar{P}_\mu^{(l)}(\bar{x}) = r^{l+1} V_\alpha(\theta) \end{aligned} \quad (14)$$

Here $\hat{\square}$ is the d'Alembertian in $n+1$ dimensions in polar coordinates, which is of course different for $\hat{P}_r^{(l)}$ and $\hat{P}_\alpha^{(l)}$. We decompose V_α into a longitudinal and a transversal part

$$V_\alpha(\theta) = \partial_\alpha \sigma(\theta) + Y_\alpha(\theta); \quad D^\alpha(\theta)Y_\alpha(\theta) = 0 \quad (15)$$

One can obtain σ by solving $D^\alpha \partial_\alpha \sigma = D^\alpha V_\alpha$ and then Y_α are given by $V_\alpha - \partial_\alpha \sigma$. Often one can directly check that in a given expression for $\partial_\alpha \sigma + Y_\alpha$ the Y_α are transversal. The $Y_\alpha(\theta)$ are the vector spherical harmonics we are interested in.

Neither ρ nor σ is a scalar spherical harmonic because neither is totally symmetric or traceless (as we shall check in examples), but linear combinations of ρ and σ yield the scalar spherical harmonics $Y^{l+1}(\theta)$ and $Y^{(l-1)}(\theta)$. A Young tableau illustrates these decompositions[4, 5]

$$\begin{array}{c} \square \otimes \square \square = \square \square \square \oplus \begin{array}{|c|c|} \hline \square & \square \\ \hline \square & \\ \hline \end{array} \oplus \square \quad (\text{for } l = 2) \\ \frac{\partial \bar{x}^\mu}{\partial \hat{y}^\nu} \times \bar{P}_\mu^{(l)} = \rho, \sigma + Y_\alpha + \rho, \sigma \end{array} \quad (16)$$

On the left-hand side the first box corresponds to the index μ of \bar{x}^μ , and it takes into account that we are considering vectors. It is multiplied by the Young tableau of the symmetric traceless polynomial which one finds in one entry of the vector. The total number of terms on the left-hand side is $(n+1)$ for \bar{x}^μ times $d_s(n, l)$ for the number of scalar spherical harmonics in one entry of the vector.

The degeneracy of the eigenvalues of the Y_α is clear from this picture: it is equal to the number of terms on the left-hand side minus the number of the two scalar spherical harmonics

$$d_v(n, l) = (n + 1)d_s(n, l) - d_s(n, l + 1) - d_s(n, l - 1) \quad (17)$$

To obtain the eigenvalues $\lambda_v(n, l)$ defined by $-\square_S Y_\alpha = \lambda_v(n, l) Y_\alpha$ we must take the transversal part of $\hat{\square} \hat{P}_\alpha^{(l)} = 0$. There are three sets of terms in $\hat{\square} \hat{P}_\alpha^{(l)}$: purely radial terms with $\partial/\partial r$ and powers of r , terms which constitute $\frac{1}{r^2} \square_S(\theta)$ for a vector, and mixed terms due to the Christoffel symbols $\hat{\Gamma}_{\beta\gamma}^r$ and $\hat{\Gamma}_{r\alpha}^\beta$. The nonvanishing connections for the metric in (5) are

$$\hat{\Gamma}_{r\alpha}^\beta = \frac{1}{r} \delta_\alpha^\beta; \quad \hat{\Gamma}_{\alpha\beta}^r = -r g_{\alpha\beta}(\theta) \quad \text{and} \quad \hat{\Gamma}_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma(\theta) \quad (18)$$

After having decomposed $\hat{\square} \hat{P}_\alpha^{(l)}$ into the three sets of terms, one may use the following identities to project out the transversal part

$$D^\alpha(\theta) \square_S(\theta) Y_\alpha(\theta) = 0; \quad \square_S \partial_\alpha \sigma = \partial_\alpha \square_S \sigma - R_{\alpha\beta}(\theta) D^\beta \sigma \quad (19)$$

There is still some tedious (but perfectly straightforward) algebra involved, but the result is very simple

$$\lambda_v(n, l) = l(n + l - 1) - 1 \quad (20)$$

Let us give some examples.

$l = 0$. For $l = 0$ the polynomial $\bar{P}_\mu^{(0)}$ is a constant, and $V_\alpha = \partial_\alpha \bar{x}^\mu$. Hence for $l = 0$ there is only a longitudinal part, but no Y_α .

$l = 1$. $\bar{P}_\rho^{(1)} = \bar{x}^\nu$ if $\rho = \mu$, and zero otherwise. Then $\hat{P}_r^{(1)} = \frac{\partial \bar{x}^\mu}{\partial r} \bar{x}^\nu = \frac{\bar{x}^\mu \bar{x}^\nu}{r}$ and $\hat{P}_\alpha^{(1)} = \frac{\partial \bar{x}^\mu}{\partial \theta^\alpha} \bar{x}^\nu$. The decomposition of $\hat{P}_\alpha^{(1)} = r^2 V_\alpha$ into a total derivative and a transversal part according to (15) is as follows

$$\hat{P}_\alpha^{(1)}(r, \theta) = \frac{\partial}{\partial \theta^\alpha} \left(\frac{1}{2} \bar{x}^\mu \bar{x}^\nu \right) + \frac{1}{2} \left(\frac{\partial \bar{x}^\mu}{\partial \theta^\alpha} \bar{x}^\nu - \frac{\partial \bar{x}^\nu}{\partial \theta^\alpha} \bar{x}^\mu \right)$$

so

$$\begin{aligned} \rho(\theta) &= \frac{\bar{x}^\mu \bar{x}^\nu}{r^2}; \quad \sigma = \frac{1}{2} \frac{\bar{x}^\mu \bar{x}^\nu}{r^2} + c \delta^{\mu\nu}; \\ Y_\alpha(\theta) &= \frac{1}{2r^2} (\partial_\alpha \bar{x}^\mu \bar{x}^\nu - \partial_\alpha \bar{x}^\nu \bar{x}^\mu) \end{aligned} \quad (21)$$

where c is an integration constant. (To prove that $D^\alpha Y_\alpha = 0$, note that $D^\alpha \partial_\alpha \bar{x}^\rho$ is proportional to \bar{x}^ρ because \bar{x}^ρ is a scalar spherical harmonic, the harmonic with $l = 1$.) Hence the two scalar harmonics that are linear combinations of ρ and σ , and the transversal vector harmonic Y_α are given by

$$\begin{aligned} \square \square &= \left(1 + \frac{1}{2c} \frac{1}{n+1} \right) \rho + \left(-\frac{1}{c} \frac{1}{n+1} \right) \sigma \\ \bullet &= \frac{1}{c} \left(\sigma - \frac{1}{2} \rho \right); \quad Y_\alpha = \frac{\nu}{\mu} = \frac{1}{2r^2} (\partial_\alpha \bar{x}^\mu \bar{x}^\nu - \partial_\alpha \bar{x}^\nu \bar{x}^\mu) \end{aligned} \quad (22)$$

These Y_α are the Killing vectors on S_n , of which there are $\frac{1}{2}(n+1)n$ (namely $\frac{1}{2}n(n-1)$ rotations and n translations). They satisfy the Killing equation $D_\alpha K_\beta + D_\beta K_\alpha = 0$, as is clear by substituting (22). Acting with D^α on this equation one confirms that $\lambda_v(n, l=1) = n-1$.¹

$l = 2$. This is fun. We choose $\bar{P}_\sigma^{(2)} = \bar{x}^\nu \bar{x}^\rho - \frac{1}{n+1} \delta^{\nu\rho} \bar{x}^2$ for $\sigma = \mu$. Then, defining $\bar{P}^{\nu\rho} = \bar{x}^\nu \bar{x}^\rho - \frac{1}{n+1} \delta^{\nu\rho} \bar{x}^2$ we obtain

$$\rho(\theta) = \frac{\bar{x}^\mu}{r^3} \bar{P}^{\nu\rho}; \quad V_\alpha(\theta) = \frac{1}{r^3} \frac{\partial \bar{x}^\mu}{\partial \theta^\alpha} \bar{P}^{\nu\rho} \quad (23)$$

The decomposition of V_α into a total derivative and a transversal part requires more work

$$\begin{aligned} r^3 \sigma(\theta) &= \frac{1}{3} \bar{x}^\mu \bar{x}^\nu \bar{x}^\rho - \frac{1}{3n} (\delta^{\mu\nu} \bar{x}^\rho + \delta^{\mu\rho} \bar{x}^\nu) \bar{x}^2 + \left(\frac{2}{3n} - \frac{1}{n+1} \right) \delta^{\nu\rho} \bar{x}^\mu \bar{x}^2 \\ r^3 Y_\alpha(\theta) &= \begin{array}{|c|c|c|} \hline \nu & & \rho \\ \hline \mu & & \\ \hline \end{array} = \frac{2}{3} \partial_\alpha \bar{x}^\mu \bar{Q}^{\nu\rho} - \frac{1}{3} \partial_\alpha \bar{x}^\nu \bar{Q}^{\mu\rho} - \frac{1}{3} \partial_\alpha \bar{x}^\rho \bar{Q}^{\mu\nu} \end{aligned} \quad (24)$$

where $\bar{Q}^{\mu\nu} = \bar{x}^\mu \bar{x}^\nu - \frac{1}{n} \delta^{\mu\nu} \bar{x}^2$. Note that there is no integration constant in σ possible because σ has an odd number (3) of indices. To prove that this is the correct decomposition, note that $\partial_\alpha \sigma + Y_\alpha = V_\alpha$ and $D^\alpha Y_\alpha = 0$.² Note that Y_α has the symmetries of the Young tableau: first antisymmetrize in $\mu\nu$, then symmetrize in $\nu\rho$. It is also traceless in all 3 pairs of indices. Referring to (16) we find

$$\begin{aligned} \begin{array}{|c|c|c|} \hline & & \\ \hline \end{array} &\sim \rho + n\sigma = \frac{n+3}{3} \left[\bar{x}^\mu \bar{x}^\nu \bar{x}^\rho - \frac{1}{n+3} (\delta^{\mu\nu} \bar{x}^\rho + \delta^{\mu\rho} \bar{x}^\nu + \delta^{\nu\rho} \bar{x}^\mu) \bar{x}^2 \right] \\ \begin{array}{|c|} \hline \\ \hline \end{array} &\sim \rho - 3\sigma = \frac{1}{n} (\delta^{\mu\nu} \bar{x}^\rho + \delta^{\mu\rho} \bar{x}^\nu) \bar{x}^2 - 3 \left(\frac{2}{3n} - \frac{1}{n+1} \right) \delta^{\nu\rho} \bar{x}^\mu \bar{x}^2 \end{aligned} \quad (25)$$

We can now exhibit the spectrum of transversal vector fields on S_5 in a similar figure as for the scalars

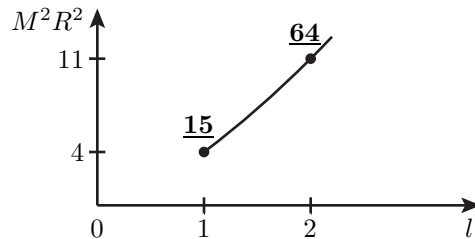


Fig. 2. S_5 , vectors (cf. Figure 1 of KRN)

The number of Killing vectors on S_5 is 15, namely 5 for translations and 10 for rotations.

¹ Use that $R_{\alpha\beta\gamma\delta}(\Gamma(\theta)) = -g_{\alpha\gamma}g_{\beta\delta} + g_{\alpha\delta}g_{\beta\gamma}$ on S_n , hence $R_{\alpha\gamma} \equiv R_{\alpha\beta\gamma}{}^\beta = -(n-1)g_{\alpha\gamma}$. (In our conventions, the scalar curvature $R = R_{\alpha\beta}g^{\alpha\beta}$ of S_n is negative.)

² Use $D^\alpha \partial_\alpha \bar{x}^\mu = -n\bar{x}^\mu$, and $\frac{1}{r^2} (\partial_\alpha \bar{x}^\mu) D^\alpha \bar{x}^\nu = \delta^{\mu\nu} - \frac{1}{r^2} \bar{x}^\mu \bar{x}^\nu$ (because $\frac{1}{r^2} \partial_\alpha \bar{x}^\mu D^\alpha \bar{x}^\nu = \delta^{\rho\sigma} \partial_\rho \bar{x}^\mu \partial_\sigma \bar{x}^\nu - \partial_r \bar{x}^\mu \partial_r \bar{x}^\nu$).

A. The vector spherical harmonics in electromagnetism.

In Jackson's book on Electromagnetism[6] one can find "multipole expansions" of the electric and magnetic fields \vec{E} and \vec{B} with time dependence $e^{-i\omega t}$ in a source-free region. These fields are decomposed into radial parts and angular parts, and we want to clarify in this section the relation of these angular parts to the vector spherical harmonics $Y_\alpha(\theta)$ satisfying $D^\alpha Y_\alpha = 0$ which we discussed before. There are differences

- (i) the wave equations for the electric and magnetic fields are $(\nabla^2 + k^2)\vec{E} = 0$ and $(\nabla^2 + k^2)\vec{B} = 0$ where $k = \omega/c$, instead of $\nabla^2 \vec{E} = \nabla^2 \vec{B} = 0$;
- (ii) the Maxwell equations impose constraints on \vec{E} and \vec{B} : $\text{div} \vec{E} = \text{div} \vec{B} = 0$ and $\text{curl} \vec{E} = ik\vec{B}$, $\text{curl} \vec{B} = -ik\vec{E}$.

However, one would expect that the angular parts of vectors on S_2 in both approaches are related.

The multipole expansion of \vec{E} is given by[6]

$$\vec{E} = \sum_{l,m} \left[a_M(l,m) g_l(kr) \frac{1}{\sqrt{l(l+1)}} \vec{L} Y_{lm}(\theta, \varphi) + \frac{i}{k} a_E(l,m) \text{curl} \left(f_l(kr) \frac{1}{\sqrt{l(l+1)}} \vec{L} Y_{lm}(\theta, \varphi) \right) \right] \quad (26)$$

and a similar expression for \vec{B} . The functions $f_l(kr)$ and $g_l(kr)$ are spherical Bessel functions, $a_E(l,m)$ and $a_M(l,m)$ are arbitrary constants, and $\vec{L} = -i\vec{r} \times \vec{\nabla}$. Clearly, these vectors are given in Cartesian coordinates.

The first term in \vec{E} is transversal due to $\vec{r} \cdot \vec{L} = 0$, but the second term contains both a longitudinal part (in the near zone $\vec{E} \rightarrow -\vec{\nabla} Y_{lm}/r^{l+1}$) and a transversal part (in the radiation zone both \vec{E} and \vec{B} become transversal). The same holds for \vec{B} . Thus the $\vec{L} Y_{lm}$, when written in polar coordinates, are equal to our $V_\alpha(r, \theta)$. Furthermore, since $\vec{\nabla} \cdot \vec{L} = 0$, these $V_\alpha(r, \theta)$ satisfy $D^\alpha V_\alpha = 0$, and hence **the $\vec{L} Y_{lm}$ are the transversal vector harmonics $Y_\alpha(\theta)$ we have constructed before, but in Cartesian coordinates.**

Let us check these statements in a few examples.

Y_{1m} . Taking Y_{lm} to be given by x/r , we find

$$\vec{L} \frac{x}{r} = \begin{pmatrix} 0 \\ z/r \\ -y/r \end{pmatrix} \quad (27)$$

and in polar coordinates (r, θ^α) with $\theta^\alpha = (\theta, \varphi)$

$$\left(\vec{L} \frac{x}{r} \right)_r \equiv \frac{\vec{r}}{r} \cdot \vec{L} \frac{x}{r} = 0, \quad \left(\vec{L} \frac{x}{r} \right)_\alpha = \frac{\partial y}{\partial \theta^\alpha} \frac{z}{r} - \frac{\partial z}{\partial \theta^\alpha} \frac{y}{r} \quad (28)$$

The vectors $\frac{1}{r^2} \left(\frac{\partial y}{\partial \theta^\alpha} z - \frac{\partial z}{\partial \theta^\alpha} y \right)$ are Killing vectors on S_2 as we have shown previously, hence they are indeed the Y_α which are obtained from $\bar{P}_\mu(\bar{x}) = \bar{x}^\nu$.

\mathbf{Y}_{2m} . Next consider $Y_{lm} = xy/r^2$. Then one finds

$$\begin{aligned}\vec{L} Y_{lm} &= \frac{1}{r^2} \begin{pmatrix} -xz \\ yz \\ x^2 - y^2 \end{pmatrix}, \quad (\vec{L} Y_{lm})_r = 0 \\ (\vec{L} Y_{lm})_\alpha &= \frac{\partial x}{\partial \theta^\alpha}(-xz) + \frac{\partial y}{\partial \theta^\alpha}(yz) + \frac{\partial z}{\partial \theta^\alpha}(x^2 - y^2)\end{aligned}\quad (29)$$

In order to compare with the Y_α we obtained from polynomials $\bar{P}_\mu(\bar{x})$, we consider in turn the polynomials $-xz$, then yz , and finally $x^2 - \frac{1}{n}\bar{r}^2 = \frac{1}{2}(x^2 - y^2 - z^2)$ and $-(y^2 - \frac{1}{n}\bar{r}^2) = -\frac{1}{2}(y^2 - x^2 - z^2)$. According to (24) they give the following vector harmonics Y_α

$$\begin{aligned}-xz &: \frac{2}{3} \frac{\partial x}{\partial \theta^\alpha}(-xz) - \frac{1}{3} \frac{\partial x}{\partial \theta^\alpha}(-xz) - \frac{1}{3} \frac{\partial z}{\partial \theta^\alpha} \left(-\frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 \right) \\ yz &: \frac{2}{3} \frac{\partial y}{\partial \theta^\alpha}(yz) - \frac{1}{3} \frac{\partial y}{\partial \theta^\alpha}(yz) + \frac{1}{3} \frac{\partial z}{\partial \theta^\alpha} \left(\frac{1}{2}y^2 - \frac{1}{2}x^2 - \frac{1}{2}z^2 \right) \\ \frac{1}{2}(x^2 - y^2 - z^2) &: \frac{2}{3} \frac{\partial z}{\partial \theta^\alpha} \left(\frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2 \right) - \frac{1}{3} \frac{\partial x}{\partial \theta^\alpha}(xz) - \frac{1}{3} \frac{\partial x}{\partial \theta^\alpha}(xz) \\ -\frac{1}{2}(y^2 - x^2 - z^2) &: \frac{2}{3} \frac{\partial z}{\partial \theta^\alpha} \left(-\frac{1}{2}y^2 + \frac{1}{2}x^2 + \frac{1}{2}z^2 \right) - \frac{1}{3} \frac{\partial y}{\partial \theta^\alpha}(yz) - \frac{1}{3} \frac{\partial y}{\partial \theta^\alpha}(yz)\end{aligned}\quad (30)$$

The sum of these terms is indeed equal to (29).

Let us next consider the terms in \vec{E} of the form $\text{curl } \vec{L}(f_l(kr)Y_{lm})$. It is useful to decompose the operator $\text{curl } \vec{L}$ as follows[6]

$$i\vec{\nabla} \times \vec{L} = \epsilon^{ijk} \nabla_j (\epsilon_{klm} x^l \nabla^m) = \vec{r} \nabla^2 - \vec{\nabla} \left(1 + r \frac{\partial}{\partial r} \right) \quad (31)$$

The functions $f_l(kr)Y_{lm}$ satisfy the wave equation $(\nabla^2 + k^2)f_l(kr)Y_{lm} = 0$, hence the second term in the multipole expansion of \vec{E} reads

$$-\frac{1}{k} a_E(l, m) \left[k^2 \vec{r} + \vec{\nabla} \left(1 + r \frac{\partial}{\partial r} \right) \right] \frac{g_l(kr)}{\sqrt{l(l+1)}} Y_{lm}(\theta, \varphi) \quad (32)$$

The first term contributes only to ρ (it lies along \vec{r}), but the second term is a gradient, hence it has a part with $\vec{\nabla} = \frac{\partial}{\partial r}$ which contributes to ρ ,³ and a part due to $\nabla = \frac{\partial}{\partial \theta^\alpha}$ which contributes to σ .

Hence the vector spherical harmonics of electromagnetism coincide with the harmonics we have discussed. In particular $\vec{L} Y_{lm}$ yields only Y_α but no $\partial_\alpha \sigma$. Note that these simple formulas only make sense on S_2 .

³ This part can be simplified using

$$\frac{\partial}{\partial r} \left(1 + r \frac{\partial}{\partial r} \right) g_l(kr) = \left(-k^2 + \frac{l(l+1)}{r} \right) g_l(kr)$$

IV. SPIN 1/2 SPHERICAL HARMONICS

The explicit construction of harmonics for scalars and vectors was discussed in references [4] and [5], but the case of spinorial harmonics was not tackled there, and turned out to be more complicated. A short conversation with Gary Gibbons at a conference in India was helpful. He had determined the spectrum of spin 1/2 harmonics on S_3 by choosing the natural diagonal vielbein field in polar coordinates, and then writing down the Dirac equation [7]. However, the form of the spinor harmonics is not determined in this way (nor was it needed for his purposes). Several discussions over the years with Martin Rocek helped me further. Eventually it became clear to me that one should introduce in addition to a change of coordinates also a local Lorentz rotation, and the problem was to determine the relation between this local Lorentz rotation and the vielbein field in the new coordinates.

We start[7] with the Dirac equation for a massless spin 1/2 field in the embedding flat $(n+1)$ -dimensional Euclidean space with Cartesian coordinates ⁴

$$\bar{\partial}\bar{\psi}(\bar{x}) \equiv \gamma^m \delta_m^\mu \frac{\partial}{\partial \bar{x}^\mu} \bar{\psi}^A(\bar{x}) = 0 \quad (33)$$

where A is the spinor index and the matrices γ^m are constant Dirac matrices in $n+1$ dimensions. The field $\bar{\psi}$ is a spinor (nonchiral if $n+1$ is even) that is homogeneous of order l in \bar{x}^μ . We construct a basis such that $\bar{\psi}(\bar{x}) = \bar{x}^{\mu_1} \dots \bar{x}^{\mu_l} \alpha + \text{“trace terms”}$, where the constant spinor α has only one nonvanishing component, and the terms denoted by “trace terms” are terms depending on $\bar{x} = \bar{x}^\mu \delta_\mu^m \gamma_m$ such that the Dirac equation holds.

For example, for $l=1$ we get

$$\bar{\psi}^{(1)}(\bar{x}) = \bar{x}^\mu \alpha - \frac{1}{n+1} \bar{x} \gamma^\mu \alpha \quad (34)$$

It is clear that $\bar{\partial}\bar{\psi}^{(1)} = 0$ since $\bar{\partial}\bar{x} = n+1$. If $n+1=3$ there are 6 such spinors for $l=1$, but the Dirac equation yields 2 constraints, hence there are 4 linearly independent $\bar{\psi}^{(1)}(\bar{x})$ on S_2 .

For $l=2$ we obtain

$$\bar{\psi}^{(2)}(\bar{x}) = \left[\bar{x}^\mu \bar{x}^\nu - \frac{1}{n+3} (\bar{x} \gamma^\mu \bar{x}^\nu + \bar{x} \gamma^\nu \bar{x}^\mu + \bar{x}^2 \delta^{\mu\nu}) \right] \alpha \quad (35)$$

which depends only on \bar{x} since $\bar{x}^2 = \bar{x} \bar{x}$. It is not only “gamma traceless” (due to the Dirac equation, see the next paragraph) but also traceless in the ordinary sense (as is obvious by taking the trace over μ and ν), as it should be since $\bar{\partial}\bar{\partial}\bar{\psi}(\bar{x}) = \bar{\square}\bar{\psi}(\bar{x})$.

One can formally write the spinor $\bar{\psi}(\bar{x})$ as

$$\bar{\psi}(\bar{x}) = \bar{x}^{\mu_1} \dots \bar{x}^{\mu_l} c^A_{\mu_1 \dots \mu_l} \quad (36)$$

where A is the spinor index. Clearly c is symmetric in $\mu_1 \dots \mu_l$ and gamma-traceless

$$(\gamma^{\mu_1})^A_B c^B_{\mu_1 \mu_2 \dots \mu_l} = 0 \quad (37)$$

⁴ Our conventions are that all γ^m are hermitian and $\gamma^m \gamma^m = I$ (no summation over m). In $n+1$ dimensions they are $2^{\lfloor \frac{n+1}{2} \rfloor} \times 2^{\lfloor \frac{n+1}{2} \rfloor}$ matrices.

Contracting with γ^{μ_2} shows that it is also traceless in the ordinary sense

$$\delta^{\mu\nu} c^A_{\mu\nu\mu_3\dots\mu_l} = 0 \quad (38)$$

We shall not use this formulation in terms of c .

The number of linearly independent spinors $\bar{\psi}^{(l)}(\bar{x})$ is equal to the number of components of α times the number of polynomials $\bar{x}_{\mu_1} \dots \bar{x}_{\mu_l}$ in $n+1$ dimensions minus the number of polynomials with one index less (to account for the Dirac equation)

$$\text{number of } \bar{\psi}^{(l)}(\bar{x}) = \left[\binom{n+l}{l} - \binom{n+l-1}{l-1} \right] 2^{\lfloor \frac{n+1}{2} \rfloor} \quad (39)$$

Next we transform to coordinates \hat{y}^ν . In practice these new coordinates are either polar coordinates (r, θ^α) or stereographic coordinates (z^α, R) , but for the time being we leave the choice open and proceed formally. The Dirac equation becomes

$$\gamma^m \left(\delta_m^\mu \frac{\partial \hat{y}^\nu}{\partial \bar{x}^\mu} \right) \frac{\partial}{\partial \hat{y}^\nu} \hat{\psi}(\hat{y}) = 0; \quad \hat{\psi}(\hat{y}) = \bar{\psi}(\bar{x}) \quad (40)$$

The composite object $\delta_m^\mu \frac{\partial \hat{y}^\nu}{\partial \bar{x}^\mu}$ is the vielbein field $E^\nu_m(\hat{y})$ in $n+1$ dimensions, and it is non-diagonal because the tangent frames are still along the axes of the \bar{x} system. When we choose polar coordinates there is a natural vielbein field that is diagonal[7] (see below), so this requires to rotate the tangent frames such that their axes become aligned along the polar coordinate axes. This motivates us to apply also in the general case a local Lorentz transformation with Lorentz matrix $\Lambda(\hat{y})$ in the spinor representation ⁵ (Λ is reducible when $n+1$ is even; we discuss the consequences later).

To implement a local Lorentz transformation by substitution we insert unity in front of $\hat{\psi}(\hat{y})$, as $\hat{\psi} = \Lambda \Lambda^{-1} \hat{\psi}$, and define

$$\psi(\hat{y}) = \Lambda^{-1}(\hat{y}) \hat{\psi}(\hat{y}) \quad (41)$$

The Dirac equation for $\psi(\hat{y})$ can then be written as follows after adding a factor Λ^{-1} in front of the equation

$$(\Lambda^{-1} \gamma^m \Lambda) \left(\delta_m^\mu \frac{\partial \hat{y}^\nu}{\partial \bar{x}^\mu} \right) \left(\frac{\partial}{\partial \hat{y}^\nu} + (\Lambda^{-1} \frac{\partial}{\partial \hat{y}^\nu} \Lambda) \right) \psi(\hat{y}) = 0 \quad (42)$$

If the finite local Lorentz transformation is given by the $(n+1) \times (n+1)$ matrix $L^m_n = (\exp \lambda)^m_n$, where λ^m_n lies in the algebra of $SO(n+1)$, then $\Lambda = \exp \frac{1}{4} \lambda^{mn} \gamma_m \gamma_n$. It follows that

$$(\Lambda^{-1} \gamma^m \Lambda) = L^m_n \gamma^n \quad (43)$$

and substituting this back into the Dirac equation yields

$$\gamma^n \left[\frac{\partial \hat{y}^\nu}{\partial \bar{x}^\mu} \delta_m^\mu L^m_n \right] \left\{ \frac{\partial}{\partial \hat{y}^\nu} + (\Lambda^{-1} \frac{\partial}{\partial \hat{y}^\nu} \Lambda) \right\} \psi(\hat{y}) = 0 \quad (44)$$

⁵ In the case of stereographic coordinates we shall first choose a suitable coset representative in the vector representation for the coset $S_n = SO(n+1)/SO(n)$. The Cartan-Maurer equations will then give us the corresponding vielbein field and from it we shall then determine the matrix Λ for this case.

The vielbein field is now given by

$$E^\nu{}_n = \frac{\partial \hat{y}^\nu}{\partial \bar{x}^\mu} \delta^\mu{}_m L^m{}_n \quad (45)$$

Given $E^\nu{}_n$ one can determine $L^m{}_n$ (as in the case of polar coordinates), or given $L^m{}_n$ one can calculate the corresponding $E^\nu{}_n$ (as in the case of stereographic coordinates).

The operator $\frac{\partial}{\partial \hat{y}^\nu} + (\Lambda^{-1} \frac{\partial}{\partial \hat{y}^\nu} \Lambda)$ is the covariant derivative for spin 1/2 spinors in $(n+1)$ dimensions. The object $(\Lambda^{-1} \frac{\partial}{\partial \hat{y}^\nu} \Lambda)$ is the spin connection in frames that are rotated with respect to the \bar{x} frames; it is, of course, pure gauge because the spin connection in the \bar{x} system vanishes. There is another way of looking at this result which will be useful when we discuss the approach with stereographic coordinates. Invert (45) to obtain $E^n{}_\nu = (L^{-1})^n{}_m \delta^\mu{}_m \frac{\partial \bar{x}^\mu}{\partial \hat{y}^\nu}$ and multiply by $d\hat{y}^\nu$. This yields the one-form equation

$$E^n = (L^{-1})^n{}_m d\bar{x}^m \quad \text{with} \quad E^n = E^n{}_\nu d\hat{y}^\nu \quad (46)$$

The spin connection Ω^{mn} which one obtains from the Cartan-Maurer equation $dE^m + \Omega^m{}_n \wedge E^n = 0$ is then $\Omega = L^{-1}dL$, and this agrees with $\Lambda^{-1}d\Lambda = \frac{1}{4}(L^{-1}dL)^{mn}\gamma_m\gamma_n$.

The Dirac equation, still in $n+1$ dimensions, can be succinctly written as

$$\hat{D}\psi = 0, \quad \hat{D} = E^\nu{}_m \gamma^m \left(\frac{\partial}{\partial \hat{y}^\nu} + \frac{1}{4} \Omega_\nu{}^{mn} \gamma_m \gamma_n \right) \quad (47)$$

Let us now first specialize to polar coordinates, and then to stereographic coordinates.

A. Polar coordinates

For definiteness we consider the case $n=3$, although all formulas hold for general n . We set

$$\begin{aligned} (ds)^2 &= d\bar{x}^2 + d\bar{y}^2 + d\bar{z}^2 + d\bar{v}^2 \\ \bar{x} &= r \cos \theta, \quad \bar{y} = r \sin \theta \cos \varphi \\ \bar{z} &= r \sin \theta \sin \varphi \cos \chi, \quad \bar{v} = r \sin \theta \sin \varphi \sin \chi \\ (ds)^2 &= dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 + r^2 \sin^2 \theta \sin^2 \varphi d\chi^2 \end{aligned} \quad (48)$$

A natural set of diagonal vielbein one-forms is then[7]

$$E^1 = dr, \quad E^2 = r d\theta, \quad E^3 = r \sin \theta d\varphi, \quad E^4 = r \sin \theta \sin \varphi d\chi \quad (49)$$

The corresponding spin connections are obtained by solving the Cartan-Maurer equation $dE^m + \Omega^{mn} \wedge E^n = 0$. The solution has the following nonvanishing components[7]

$$\begin{aligned} \Omega^{21} &= d\theta, \quad \Omega^{31} = \sin \theta d\varphi, \quad \Omega^{32} = \cos \theta d\varphi \\ \Omega^{41} &= \sin \theta \sin \varphi d\chi, \quad \Omega^{42} = \cos \theta \sin \varphi d\chi, \quad \Omega^{43} = \sin \theta \cos \varphi d\chi \end{aligned} \quad (50)$$

Actually, all we need in $n+1$ dimensions is that $\Omega_r{}^{pq} = 0$ and that $\Omega^{a1} = \frac{1}{r} E^a d\theta^a$. This is easily proven using that $E^1 = dr$ and E^a for $a \neq 1$ is proportional to r .

For $n = 3$ the Dirac equation for $\psi(\hat{y}) = \Lambda^{-1}\hat{\psi}(\hat{y})$ in polar coordinates and in tangent frames aligned along the coordinate axes reads

$$\left[\gamma^1 \partial_r + \frac{1}{4} \frac{\gamma^2}{r} (2\gamma^2 \gamma^1) + \frac{1}{4} \frac{\gamma^3}{r \sin \theta} (2 \sin \theta \gamma^3 \gamma^1) + \frac{\gamma^4}{4r \sin \theta \sin \varphi} (2 \sin \theta \sin \varphi \gamma^4 \gamma^1) + \frac{1}{r} \mathcal{D}_S(\theta) \right] \psi(\hat{y}) = 0 \quad (51)$$

We shall determine the matrix Λ later, see equation (79). The operator $\mathcal{D}_S(\theta)$ is the Dirac operator on S_n ; it depends only on the angles θ . Note that the second, third and fourth terms are equal. Generalizing to n dimensions yields

$$-\gamma^1 \left(\partial_r + \frac{n}{2} \frac{1}{r} \right) \psi(\hat{y}) = \frac{1}{r} \mathcal{D}_S(\theta) \psi(\hat{y}) \quad (52)$$

Since $\psi = \Lambda^{-1}\hat{\psi} = \Lambda^{-1}\bar{\psi}(\bar{x})$ is still homogeneous in r (because Λ does not depend on r), we have

$$\psi(r, \theta) = r^l \psi(\theta) \quad (53)$$

To obtain the eigenvalues of \mathcal{D}_S we must remove the matrix γ^1 in the Dirac equation in (52). This is achieved in different ways for $n+1$ even and $n+1$ odd.

Even $n+1$: we choose a suitable representation

$$\gamma^1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^{a+1} = \begin{pmatrix} 0 & -i\sigma^a \\ i\sigma^a & 0 \end{pmatrix} \quad \text{for } a = 1, \dots, n \quad (54)$$

where σ^a is any set of Dirac matrices in n dimensions. Setting $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$, we obtain two diagonal equations

$$\left. \begin{aligned} -i \mathcal{D}_S \psi_- &= -(l + \frac{n}{2}) \psi_- \\ +i \mathcal{D}_S \psi_+ &= -(l + \frac{n}{2}) \psi_+ \end{aligned} \right\} \quad \begin{aligned} \lambda_{spinor}(n, l) &= \pm(l + \frac{n}{2}) \\ l &= 0, 1, 2, \dots \end{aligned} \quad (55)$$

Odd $n+1$: we contract with $1 \pm i\gamma^1$ (which are *not* projection operators) using $(1 \pm i\gamma^1)\gamma^1 = \pm i(1 \mp i\gamma^1)$. Then we get[7]

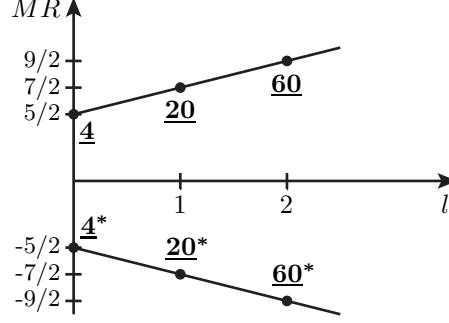
$$\mathcal{D}_S[(1 \mp i\gamma^1)\psi] = \mp i \left(l + \frac{n}{2} \right) [(1 \mp i\gamma^1)\psi] \quad (56)$$

so the same eigenvalues.

The degeneracies of these eigenvalues are the number of spinors $\bar{\psi}^{(l)}$ (or half as many if $n+1$ is even). Thus the spectrum is

$$\begin{aligned} \lambda_{spinor}(n, l) &= \pm \left(l + \frac{n}{2} \right) \\ d_{spinor}(n, l) &= \left[\binom{n+l}{l} - \binom{n+l-1}{l-1} \right] 2^{\lfloor \frac{n}{2} \rfloor} \end{aligned} \quad (57)$$

For S_5 we find the following spectrum (eigenvalues $\mathcal{D}_S \hat{\psi} = i\lambda_{spinor}(n, l)\hat{\psi}$ and degeneracy)

Fig. 3. S_5 , spin 1/2 (cf. Figure 5 of KRN)

B. Stereographic coordinates

Stereographic coordinates $\hat{y}^\nu = (z^\alpha, R)$ with $\alpha = 1, \dots, n$ and R again the radius are related to the Cartesian coordinates \bar{x}^μ with $\mu = 1, \dots, n+1$ by projecting onto a plane. Using that $\frac{\bar{x}^\alpha}{z^\alpha} = \frac{\bar{x}^{n+1} + R}{2R}$ (see the figure) one finds

$$\bar{x}^\alpha = \frac{z^\alpha 4R^2}{4R^2 + z^2}, \quad \bar{x}^{n+1} = R \frac{4R^2 - z^2}{4R^2 + z^2}$$

$$\delta^{\mu\nu} \bar{x}^\mu \bar{x}^\nu = R^2, \quad z^2 \equiv z^\alpha z^\beta \delta^{\alpha\beta}$$

(58)

Note that \bar{x}^{n+1} at the South Pole is positive. As coset representative L^m_n for the coset $S_n = SO(n+1)/SO(n)$ we take an $SO(n+1)$ matrix which maps the South Pole (the point with $\bar{x}^{n+1} = R$) to the point \bar{x}^μ

$$L^m_n = \begin{pmatrix} \delta^a_b - \frac{2z^a z^b}{4R^2 + z^2} & \frac{4Rz^a}{4R^2 + z^2} \\ \frac{-4Rz^b}{4R^2 + z^2} & \frac{4R^2 - z^2}{4R^2 + z^2} \end{pmatrix} \quad (59)$$

(For $z \rightarrow 0$ we obtain the unit matrix, and for this reason we choose $x^{n+1} = +R$ instead of $-R$ for the South Pole). We use these matrices for the local Lorentz rotation which acts on the tangent frames. We find then from the relation between vielbein fields and local Lorentz rotations in (45)

$$L^m_n E^n_\nu = \delta^m_\mu \frac{\partial \bar{x}^\mu}{\partial \hat{y}^\nu} \quad (60)$$

One can solve for E^n_ν (a rather tedious calculation but the result is easy to check) to obtain

$$E^n_\nu = \begin{pmatrix} \frac{4R^2}{4R^2 + z^2} \delta^a_\alpha & \frac{-4z^a R}{4R^2 + z^2} \\ 0 & 1 \end{pmatrix} \quad (61)$$

where $a = 1, \dots, n$ is a flat vector index on S_n and $\alpha = 1, \dots, n$ is a curved vector index on S_n . The result $E^a{}_\alpha = \delta^a{}_\alpha(4R^2)/(4R^2 + z^2)$ is well-known,⁶ but there is an off-diagonal term $E^a{}_{n+1} = -4z^a R/(4R^2 + z^2)$. We need the inverse vielbein for the Dirac equation. It is given by

$$E^\nu{}_n = \begin{pmatrix} \frac{4R^2 + z^2}{4R^2} \delta^\alpha{}_a & \frac{z^\alpha}{R} \\ 0 & 1 \end{pmatrix} \quad (62)$$

The Dirac equation in (47) becomes

$$\begin{aligned} & E^\alpha{}_a \gamma^a \left[\frac{\partial}{\partial z^\alpha} + (\Lambda^{-1} \frac{\partial}{\partial z^\alpha} \Lambda) \right] \psi(z, R) \\ & + \gamma^{n+1} \frac{z^\alpha}{R} \left[\frac{\partial}{\partial z^\alpha} + (\Lambda^{-1} \frac{\partial}{\partial z^\alpha} \Lambda) \right] \psi(z, R) \\ & + \gamma^{n+1} \left[\frac{\partial}{\partial R} + (\Lambda^{-1} \frac{\partial}{\partial R} \Lambda) \right] \psi(z, R) = 0 \end{aligned} \quad (63)$$

The connections are given by

$$(L^{-1} \frac{\partial}{\partial z^\alpha} L)^{ab} = \frac{1}{4R^2 + z^2} \begin{pmatrix} -2\delta^a{}_\alpha z^b + 2\delta^b{}_\alpha z^a & 4R\delta^a{}_\alpha \\ -4R\delta^b{}_\alpha & 0 \end{pmatrix} \quad (64)$$

Then⁷

$$\begin{aligned} (\Lambda^{-1} \frac{\partial}{\partial z^\alpha} \Lambda) &= e^a{}_\alpha \gamma_a \gamma_{n+1} + \frac{1}{4} \omega_\alpha{}^{ab} \gamma_a \gamma_b \\ e^a{}_\alpha &= \frac{2R\delta^a{}_\alpha}{4R^2 + z^2}, \quad \omega_\alpha{}^{ab} = \frac{-2\delta^a{}_\alpha z^b + 2\delta^b{}_\alpha z^a}{4R^2 + z^2} \end{aligned} \quad (66)$$

Similarly

$$\begin{aligned} L^{-1} \frac{\partial}{\partial R} L &= \begin{pmatrix} 0 & -4z^a/(4R^2 + z^2) \\ 4z^b/(4R^2 + z^2) & 0 \end{pmatrix} \\ \Lambda^{-1} \frac{\partial}{\partial R} \Lambda &= \frac{-2z^a}{4R^2 + z^2} \gamma_a \gamma_{n+1} = \frac{-2\not{z}}{4R^2 + z^2} \gamma_{n+1} \end{aligned} \quad (67)$$

⁶ In the literature one usually sets $R = 1/2$ in which case $E^a{}_\alpha = \delta^a{}_\alpha/(1 + z^2)$. We keep the dependence on R because we shall have to differentiate with respect to R later.

⁷ From $L^{-1} dL = e^a K_a^{(v)} + \frac{1}{2} \omega^{ab} H_{ab}^{(v)}$ and

$$K_a^{(v)} = \begin{pmatrix} 0 \\ \vdots \\ 2 \\ \vdots \\ 0 \dots -2 \dots 0 \end{pmatrix}, \quad H_{ab}^{(v)} = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ -1 \\ 0 \dots \dots \dots 0 \end{pmatrix} \quad (65)$$

we obtain $[K_a, K_b] = -4H_{ab}$. Then in the (reducible if $n + 1 = \text{even}$) spinor representation we get $K_a^{(s)} = \gamma_a \gamma_{n+1}$ and $H_{ab}^{(s)} = \frac{1}{2} \gamma_{ab}$.

Substituting these results into the Dirac equation yields

$$E^\alpha{}_a \gamma^a \left[D_\alpha \psi + e^b{}_\alpha \gamma_b \gamma_{n+1} \right] \psi + \gamma^{n+1} \frac{z^\alpha}{R} \left[\frac{\partial}{\partial z^\alpha} + e^a{}_\alpha \gamma_a \gamma_{n+1} \right] \psi + \gamma^{n+1} \left[\frac{\partial}{\partial R} - \frac{2\not{z}}{4R^2 + z^2} \gamma_{n+1} \right] \psi = 0 \quad (68)$$

where we used that $z^\alpha \omega_\alpha{}^{ab} = 0$ to replace $z^\alpha D_\alpha$ in the second line by $z^\alpha \partial_\alpha$. The last term in the first line cancels the last term in the second line. The dimensionless $E^\alpha{}_a$ is part of the $(n+1)$ -dimensional vielbein field, and the dimensionful $e^a{}_\alpha$ is the vielbein on the coset S_n . They are related by

$$E^\alpha{}_a = \frac{1}{2R} e^\alpha{}_a \quad \text{where} \quad e^a{}_\alpha e^a{}_\beta = \delta^\alpha{}_\beta \quad (69)$$

Using this, the Dirac equation reduces to

$$\frac{1}{2R} e^\alpha{}_a \gamma^a \left[D_\alpha^{(S)} + e^b{}_\alpha \gamma_b \gamma_{n+1} \right] \psi(z, R) = -\frac{\gamma^{n+1}}{R} \left[z^\alpha \frac{\partial}{\partial z^\alpha} + R \frac{\partial}{\partial R} \right] \psi(z, R) \quad (70)$$

The operator $z^\alpha \frac{\partial}{\partial z^\alpha} + R \frac{\partial}{\partial R}$ counts the number of \bar{x}^μ in the homogeneous polynomial $\bar{\psi}(\bar{x})$, hence

$$\not{D}_S \psi(z, R) = -\frac{\gamma^{n+1}}{R} \left[l + \frac{n}{2} \right] \psi(z, R) \quad (71)$$

where $\not{D}_S = E^\alpha{}_a \gamma^a \left(\frac{\partial}{\partial z^\alpha} + \dots \right)$ has dimension $(\text{length})^{-1}$. From here on we remove the matrix γ^{n+1} the same way as we removed γ^1 in the case of polar coordinates. Hence the spectrum in stereographic coordinates is the same as in polar coordinates, as it should of course be the case.

V. THE KRN METHOD

In KRN the spin 1/2 spherical harmonics $\Xi^{(l)\pm}(z)$ were constructed as a product of scalar spherical harmonics $Y^{(l)}(z)$ and Killing vectors $\eta^\pm(z)$,

$$\Xi^{(l)\pm} = A_\pm Y^l(\theta) \eta^\pm(\theta) + B_\pm (\not{D}_S Y^l) \eta^\pm(\theta) \quad (72)$$

The Killing spinors satisfy the equation $D_\alpha \eta^\pm(z) = \mp \frac{i}{2} c \tau_\alpha \eta^\pm$, with $c = 1/R$, and requiring that this ansatz satisfies $\not{D}_S \Xi^{(l)\pm} = i \lambda_\pm \Xi^{(l)\pm}$ yields the equations

$$A_\pm \pm \frac{i}{2} c B_\pm (n-2) = i \lambda_\pm B_\pm$$

$$A_\pm \left(\mp \frac{i}{2} n c \right) - B_\pm c^2 l (l+n-1) = i \lambda_\pm A_\pm \quad (73)$$

where we used $-\square_S Y^l = c^2 l (l+n-1) Y^l(\theta)$. Requiring that the determinant of the matrix of this system vanishes, one finds a quadratic equation for λ_\pm , and two solutions for λ_+ , and

two for λ_-

$$\begin{aligned}\lambda_+ &= -\frac{c}{2} \pm c \left(l + \frac{n-1}{2} \right) = \begin{cases} +c \left(l + \frac{n-2}{2} \right) \\ -c \left(l + \frac{n}{2} \right) \end{cases} \\ \lambda_- &= +\frac{c}{2} \pm c \left(l + \frac{n-1}{2} \right) = \begin{cases} +c \left(l + \frac{n}{2} \right) \\ -c \left(l + \frac{n-2}{2} \right) \end{cases}\end{aligned}\quad (74)$$

The corresponding eigenspinors on S_5 are given by

$$\left. \begin{aligned} \not{D} \left[c l Y^l \eta^+ - i(\not{D} Y^l) \eta^+ \right] &= i c \left(l + \frac{3}{2} \right) [\text{same}] \\ \not{D} \left[c(l+4) Y^l \eta^+ + i(\not{D} Y^l) \eta^+ \right] &= -i c \left(l + \frac{5}{2} \right) [\text{same}] \end{aligned} \right\} \quad \text{for } \lambda_+$$

$$\left. \begin{aligned} \not{D} \left[c(l+4) Y^l \eta^- - i(\not{D} Y^l) \eta^- \right] &= i c \left(l + \frac{5}{2} \right) [\text{same}] \\ \not{D} \left[c l Y^l \eta^- + i(\not{D} Y^l) \eta^- \right] &= -i c \left(l + \frac{3}{2} \right) [\text{same}] \end{aligned} \right\} \quad \text{for } \lambda_- \quad (75)$$

The eigenfunctions corresponding to $l + 3/2$ for $l = 0$ clearly vanish, hence all eigenvalues are given by $\lambda(n, l) = \pm(l + n/2)$ with $l = 0, 1, 2, \dots$. For $l = 0$ there are two eigenfunctions with the eigenvalue $\lambda = n/2 = 5/2$

$$\begin{aligned} \not{D} \left[Y^1 \eta^+ - i(\not{D} Y^1) \eta^+ \right] &= i c \frac{5}{2} \left[Y^1 \eta^+ - i(\not{D} Y^1) \eta^+ \right] \\ \not{D} \left[4 Y^0 \eta^- \right] &= i c \frac{5}{2} \left[4 Y^0 \eta^- \right] \end{aligned}\quad (76)$$

They are compatible if both spinors are proportional

$$Y^1 \eta^+ - i(\not{D} Y^1) \eta^+ = \eta^- \quad (77)$$

We shall check this relation later. Similarly, the two solutions for $\lambda = -n/2$ are compatible if $Y^1 \eta^- + i(\not{D} Y^1) \eta^-$ is proportional to η^+ . Substituting this relation into (77) leads to a constraint on Y^1 , namely

$$(Y^1)^2 + (D_\alpha Y^1)(D^\alpha Y^1) = \text{constant} \quad (78)$$

Given that $Y^{(1)}$ is equal to \bar{x}^μ/r , it is easy to check that this relation holds.

The degeneracy of the $l = 0$ spinor harmonics $\eta^+(\theta)$ and $\eta^-(\theta)$ on S_5 is 4 since one can choose η^\pm at $\theta = 0$ at will. This agrees with (57). For $l = 1$ the degeneracy seems to be 4×6 because in $Y^1 \eta^+$ there are six ways to choose Y^1 and four for η^+ . However, the Dirac equation subtracts one spinor, hence the degeneracy for $l = 1$ is 4×5 , again in agreement with (57). We have now a good idea about the structure of the spinorial harmonics but we still have no explicit expressions for η^\pm .

At this point a puzzle arises: in the expressions we obtained in (41) for the spin 1/2 spherical harmonics we did not find derivatives of Y^l , but in (72) they are needed to find solutions to the eigenvalue equations. How can these two expressions be equal?

To solve this problem we return to the embedding method. To obtain explicit expressions for the spinor harmonics themselves we need to determine the matrix Λ . We consider

first polar coordinates; later we discuss the stereographic coordinates. Recall the relation $E^\nu_n = \frac{\partial \hat{y}^\nu}{\partial \bar{x}^\mu} \delta^\mu_m L^m_n$. From it one can solve for L^m_n , namely

$$L^m_n = \delta^m_\mu \frac{\partial \bar{x}^\mu}{\partial \hat{y}^\nu} E^\nu_n \quad (79)$$

Using the diagonal vielbein of the previous section, the result for L^m_n on S_3 is

$$L^m_n = \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta \cos \varphi & \cos \theta \cos \varphi & -\sin \varphi & 0 \\ \sin \theta \sin \varphi \cos \chi & \cos \theta \sin \varphi \cos \chi & \cos \varphi \cos \chi & -\sin \chi \\ \sin \theta \sin \varphi \sin \chi & \cos \theta \sin \varphi \sin \chi & \cos \varphi \sin \chi & \cos \chi \end{pmatrix} \quad (80)$$

It can be written as the product of a rotation in the xy plane, followed by a rotation in the yz plane, etc.

$$\begin{aligned} L^m_n &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \chi & -\sin \chi \\ 0 & 0 & \sin \chi & \cos \chi \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi & 0 \\ 0 & \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta & 0 & 0 \\ \sin \theta & \cos \theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= (\exp \chi L_{34})(\exp \varphi L_{23})(\exp \theta L_{12}) \text{ with } L_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ for } i < j \end{aligned} \quad (81)$$

As one may check, the first rotation aligns the x -axis along the radius, the second rotation aligns the y -axis in the θ -direction, etc. Hence the matrix Λ^{-1} in (41) is given by

$$\Lambda^{-1} = e^{\frac{1}{2}\theta\gamma^1\gamma^2} e^{\frac{1}{2}\varphi\gamma^2\gamma^3} e^{\frac{1}{2}\chi\gamma^3\gamma^4} \quad (82)$$

The spinor harmonics are then given in $n+1$ =odd dimensions by multiplication by $(1 \pm i\gamma^1)$, and in $n+1$ =even dimensions by taking the chiral and antichiral parts.

Let us determine the first few spinor harmonics explicitly.

$l = 0$. The $l = 0$ case yields Killing spinors. One starts from $\bar{\psi} = \hat{\psi} = \alpha = \text{constant}$, and obtains $\psi = \Lambda^{-1}\alpha$. On S_2 we have $\Lambda^{-1} = \exp \frac{1}{2}\theta\gamma^1\gamma^2 \exp \frac{1}{2}\varphi\gamma^2\gamma^3$. Choosing $\gamma^1 = \sigma^3$, $\gamma^2 = \sigma^1$ and $\gamma^3 = \sigma^2$ where σ^j are the Pauli matrices, we get 4 Killing spinors after multiplication by $(1 \pm i\gamma^1)$

$$\begin{aligned} \eta_I^+ &= e^{i\varphi/2} \begin{pmatrix} \cos \frac{\theta}{2}(1+i) \\ -\sin \frac{\theta}{2}(1-i) \end{pmatrix}, & \eta_{II}^+ &= e^{i\varphi/2} \begin{pmatrix} \cos \frac{\theta}{2}(1-i) \\ -\sin \frac{\theta}{2}(1+i) \end{pmatrix} \\ \eta_I^- &= e^{-i\varphi/2} \begin{pmatrix} \sin \frac{\theta}{2}(1+i) \\ \cos \frac{\theta}{2}(1-i) \end{pmatrix}, & \eta_{II}^- &= e^{-i\varphi/2} \begin{pmatrix} \sin \frac{\theta}{2}(1-i) \\ \cos \frac{\theta}{2}(1+i) \end{pmatrix} \end{aligned} \quad (83)$$

One may check that they satisfy the Killing spinor equations $D_\alpha \eta^\pm = \pm \frac{i}{2} e^a_\alpha \gamma_a \eta^\pm$, which read in more explicit form

$$\partial_\theta \eta^\pm = \pm \frac{i}{2} \sigma_1 \eta^\pm, \quad \left(\partial_\varphi \mp \frac{i}{2} \cos \theta \sigma^3 \right) \eta^\pm = \pm \frac{i}{2} \sin \theta \sigma^2 \eta^\pm \quad (84)$$

$l = 1$. For $l = 1$ we have $\bar{\psi}^{(l=1)}(\bar{x}) = [\bar{x}^\mu - \frac{1}{n+1} \bar{x}^\nu \gamma^\nu \gamma^\mu] \alpha$ with constant γ^ν , γ^μ and α . We find then for $\psi = \Lambda^{-1} \hat{\psi}$

$$\psi^{(l=1)} = \bar{x}^\mu \Lambda^{-1} \alpha - \frac{1}{n+1} \bar{x}^\nu (\Lambda^{-1} \gamma^\nu \Lambda) (\Lambda^{-1} \gamma^\mu \Lambda) (\Lambda^{-1} \alpha) \quad (85)$$

The factor $\bar{x}^\nu \Lambda^{-1} \gamma^\nu \Lambda = \bar{x}^\nu \delta_{\nu n} L^n_m \gamma^m$ is only nonvanishing if $m = 1$, see (80), hence this factor yields $r\gamma^1$. The factor $\Lambda^{-1} \gamma^\mu \Lambda$ can also be simplified if one uses $\Lambda^{-1} \gamma^\mu \Lambda = L^\mu_n \gamma^n$ and $L^\mu_n = \frac{\partial \bar{x}^\mu}{\partial \hat{y}^\nu} E^\nu_n \gamma^n$. One finds then

$$\Lambda^{-1} \gamma^\mu \Lambda = \frac{\partial \bar{x}^\mu}{\partial \hat{y}^\nu} E^\nu_n \gamma^n = \frac{\bar{x}^\mu}{r} \gamma^1 + \frac{1}{r} \mathcal{P}_S \bar{x}^\mu \quad (86)$$

where we define $\mathcal{P}_S \bar{x}^\mu = \frac{\partial \bar{x}^\mu}{\partial \theta^\alpha} \frac{1}{r} \gamma^\alpha e^\alpha_a(\theta)$. Hence

$$\psi^{(l=1)} = \left[\bar{x}^\mu - \frac{1}{n+1} r \gamma^1 \left(\frac{\bar{x}^\mu}{r} \gamma^1 + \frac{1}{r} \mathcal{P}_S(\theta) \bar{x}^\mu \right) \right] \alpha \quad (87)$$

After multiplication by $(1 \pm i\gamma^1)$ we arrive at

$$\begin{aligned} \Xi^{l=1, \pm} &= \left[\frac{n}{n+1} \bar{x}^\mu \mp \frac{i}{n+1} (\mathcal{P}_S \bar{x}^\mu) \right] (1 \pm i) \alpha \\ &= \frac{1}{n+1} \left[n Y^1 \mp i (\mathcal{P}_S Y^1) \right] \eta^\pm \end{aligned} \quad (88)$$

where $Y^1 = \bar{x}^\mu$. This agrees with KRN, eqs. (3.9) and (3.10) (except for the $+$ sign in (3.10) which should clearly be a $-$ sign).

So this explains the terms with $(\mathcal{P} Y^l)$ in the ansatz of KRN: they are due to the Lorentz matrix L in $\Lambda^{-1} \gamma^m \Lambda$.

VI. CONCLUSION

We have shown how to obtain the spectra, but also the spherical harmonics themselves, for spin 0, 1, 1/2 on S_n by starting in a flat Euclidean $(n+1)$ -dimensional embedding space with Cartesian coordinates \bar{x}^μ , and then transforming to general coordinates on S_n and, for spin 1/2, to general tangent frames. We worked out in particular the cases of polar coordinates and stereographic coordinates, and we determined the local Lorentz transformations which bring the tangent frames in their final orientation. The cases for spin 3/2, spin 2, and antisymmetric tensor fields are treated the same way, but lack of space does not permit to include details of these derivations. With this information one can easily reproduce the figures and tables in KRN.

We also showed that the vector spherical harmonics in Cartesian coordinates in Jackson's book on Electromagnetism are the same as those we obtained after we transformed to polar coordinates. We explained that the derivatives in the expressions for the spinor harmonics in KRN are due to the local Lorentz rotations of the tangent frames.

One can obtain the spectra for any coset manifold G/H from only group theory, without having to know the explicit form of the spherical harmonics. One begins with a coset representative $L(z)$. Then $L^{-1} dL$, where d is the exterior derivative, lies in the Lie algebra, so $L^{-1} dL = e^a K_a + \omega^i H_i$, where K_a are the coset generators, H_i the subgroup generators, e^a the coset vielbein one-form, and ω^i the subgroup connection one-form. Next define $Y = L^{-1}$, and use $L^{-1} dL = -dY Y^{-1}$ to obtain $(d + \omega^i H_i) Y = -e^a K_a Y$. Introducing $\partial_a = e_a^\alpha \partial_\alpha$ where e_a^α is the inverse of e_α^a in $e^a = dz^\alpha e_\alpha^a$, we obtain

$$D_a Y = -K_a Y, \quad D_a = \partial_a + \omega_a^i H_i \quad (89)$$

On symmetric coset manifolds coset generators H_i span a subgroup of $SO(n)$ and ω^i is the spin connection. (A reductive coset manifold satisfies $[H_i, K_a] \subset K$, and a symmetric coset manifold satisfies in addition $[K_a, K_b] \subset H$.) Now act again with D_b

$$D_b D_a Y = -[\omega_b^i H_i, K_a] Y - K_a D_b Y \quad (90)$$

Moving the first term on the right-hand side to the left-hand side, it completes the derivative D_b to a derivative \mathcal{D}_b where \mathcal{D}_b contains also a term with the spin connection which acts on the indices a of D_a

$$\mathcal{D}_b D_a Y = K_a K_b Y \quad (91)$$

Finally take the trace to obtain $\square Y = \sum_a (K_a)^2 Y$. By writing $\sum_a (K_a)^2$ as a sum over all generators of the group G minus a sum over the generators of the subgroup H , one finds the eigenvalues of the d'Alembertian in terms of two quadratic Casimir generators

$$\square Y = \left(C_2(G) - C_2(H) \right) Y \quad (92)$$

This approach is worked out in detail in the remarkable 3-volume textbook of Castellani, D'Auria and Fré in reference [2]. Each spherical harmonic corresponds to a Young tableau of the group G . The degeneracy of the eigenvalues is the dimension of the Young tableau. The values of the Casimir operators and their degeneracy can for example be found in reference [8].

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